Solution of the 2D inviscid Burgers equation using a multi-directional upwind scheme

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Introduction

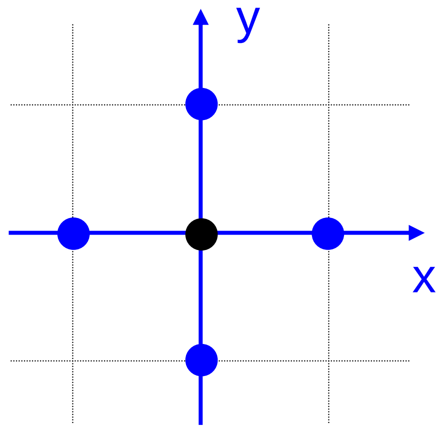
Multi-directional finite-difference scheme is one of many ideas of Prof. Kunio Kuwahara.

He showed its usefulness by successfully performing many flow simulations.

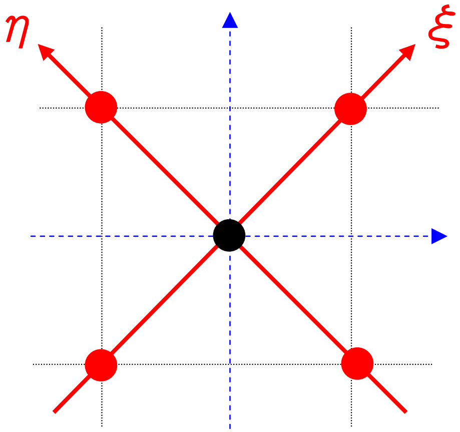
In this paper, we clearly show the effect of multi-directional finite-difference scheme when solving the inviscid Burgers equation.

Multi-directional finite-difference scheme

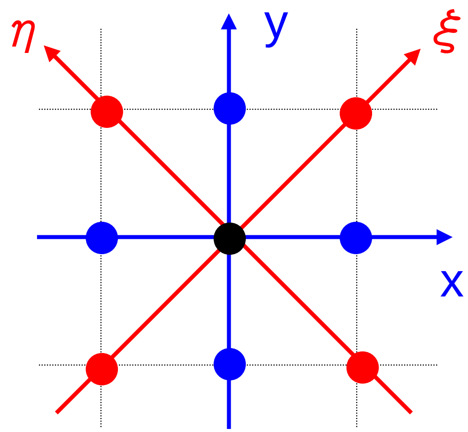
To approximate the derivatives,
grid points in regular coordinates and diagonal coordinates are used.



Grid points in regular coordinates



Grid points in diagonal coordinates



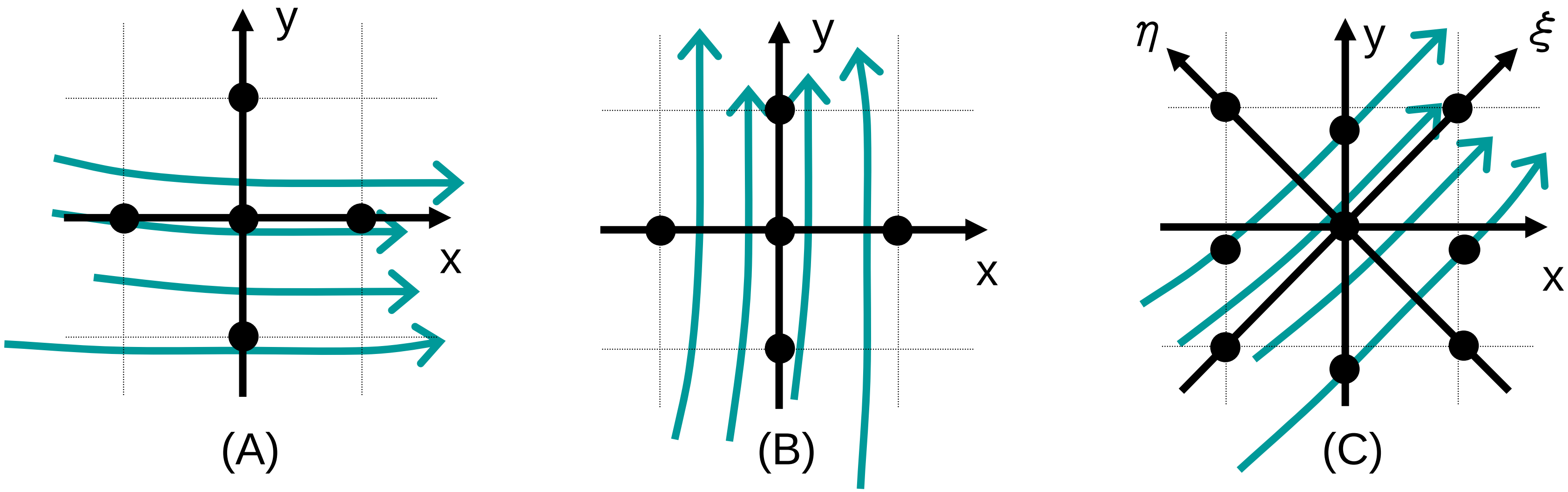
Grid points
in multi-directional coordinates

2D Inviscid Burgers Equation

2D inviscid Burgers equation is numerically solved using finite-difference method.

$$\frac{\partial \mathbf{u}}{\partial t} + \boxed{u\frac{\partial \mathbf{u}}{\partial x} + v\frac{\partial \mathbf{u}}{\partial y}} = \mathbf{0} \qquad \left(\mathbf{u} = (u, v)\right)$$

Advection terms

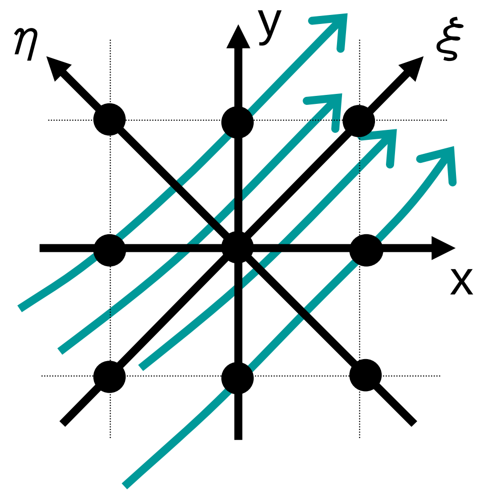


In general, flow direction is not always parallel to a coordinate line, as shown in Figures (A) and (B). We can overcome this problem using the multi-directional scheme shown in Figure (C).

Multi-directional finite-difference approximation

$$\frac{\partial \mathbf{u}}{\partial t} + \boxed{u\frac{\partial \mathbf{u}}{\partial x} + v\frac{\partial \mathbf{u}}{\partial y}} = \mathbf{0} \qquad \left(\mathbf{u} = (u, v)\right)$$

Advection terms



- Advection terms

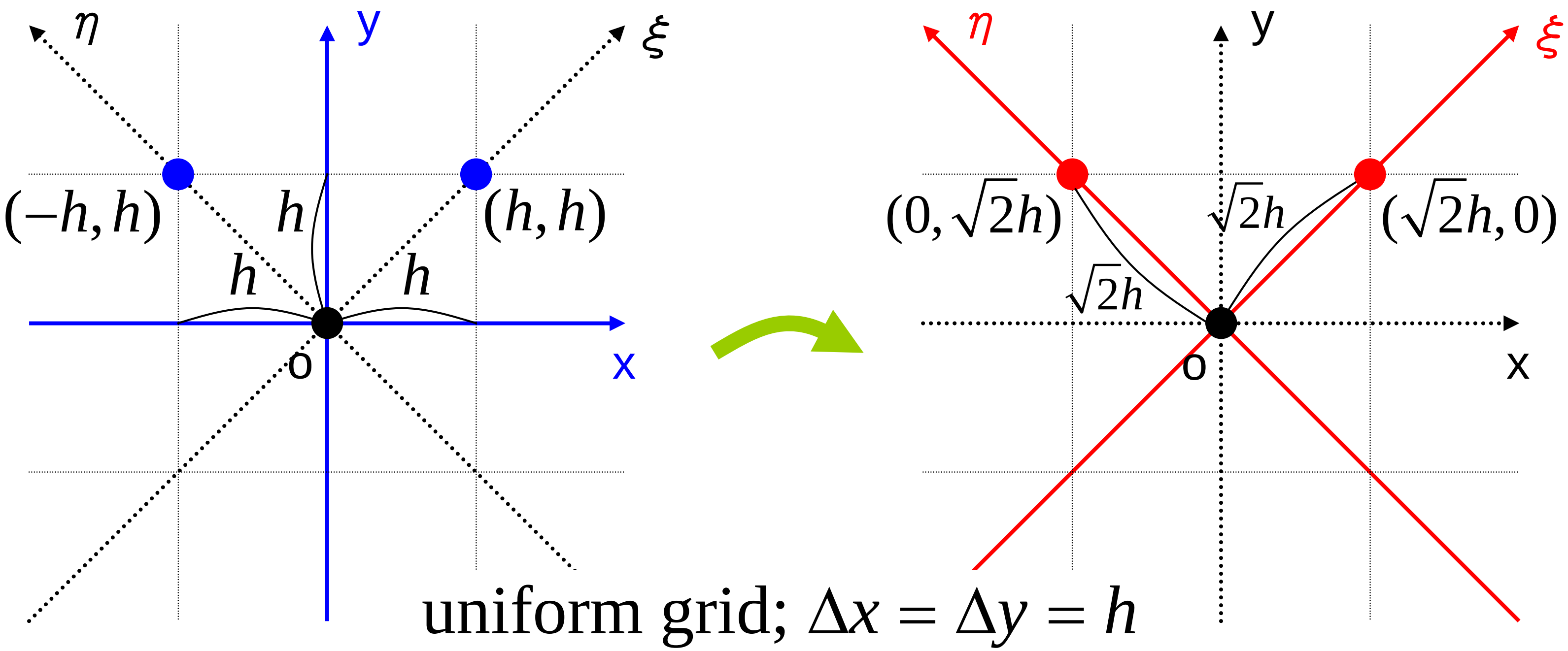
  … Third-order multi-directional upwind scheme.

- Time integration

  … Second-order Crank-Nicolson implicit scheme.

Multi-directional finite-difference representation

- Coordinate transformation into diagonal coordinates



(x,y) regular coordinates

(ξ , η) diagonal coordinates

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad J = 1$$

Multi-directional finite-difference representation

- Coordinate transformation into diagonal coordinates

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \ \xi_x = \frac{1}{\sqrt{2}}, \ \xi_y = \frac{1}{\sqrt{2}}, \ \eta_x = -\frac{1}{\sqrt{2}}, \ \eta_y = \frac{1}{\sqrt{2}}$$

Advection terms: $u\frac{\partial \mathbf{u}}{\partial x} + v\frac{\partial \mathbf{u}}{\partial y}$

$$= u\left(\xi_x \frac{\partial \mathbf{u}}{\partial \xi} + \eta_x \frac{\partial \mathbf{u}}{\partial \eta} \right) + v\left(\xi_y \frac{\partial \mathbf{u}}{\partial \xi} + \eta_y \frac{\partial \mathbf{u}}{\partial \eta} \right) = \left(\xi_x u + \xi_y v \right)\frac{\partial \mathbf{u}}{\partial \xi} + \left(\eta_x u + \eta_y v \right)\frac{\partial \mathbf{u}}{\partial \eta}$$

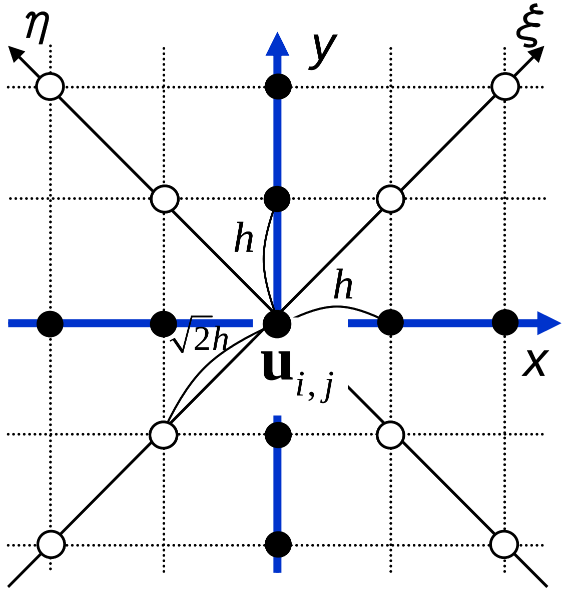
$$= \frac{1}{\sqrt{2}}\left((u+v)\frac{\partial \mathbf{u}}{\partial \xi} + (-u+v)\frac{\partial \mathbf{u}}{\partial \eta} \right)$$

$u\frac{\partial \mathbf{u}}{\partial x} + v\frac{\partial \mathbf{u}}{\partial y} \sim D_{xy}$: Finite-difference representation in regular coordinates.

$\frac{1}{\sqrt{2}}\left((u+v)\frac{\partial \mathbf{u}}{\partial \xi} + (-u+v)\frac{\partial \mathbf{u}}{\partial \eta} \right) \sim D_{\xi\eta}$: Finite-difference representation in diagonal coordinates.

Multi-directional finite-difference representation

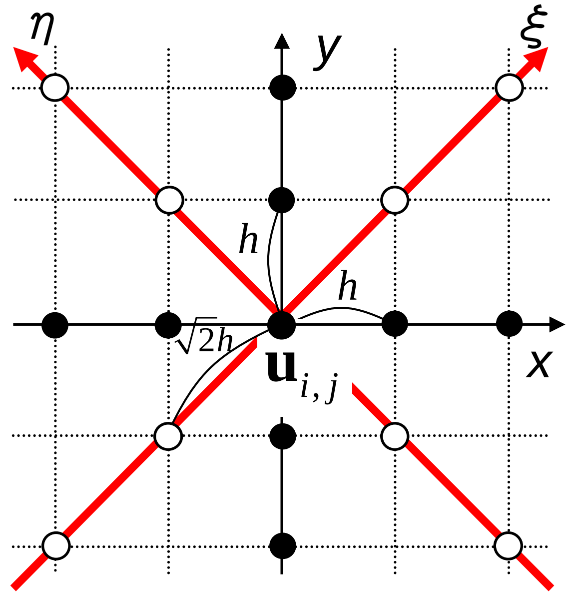
- Discretization on the regular grid using a third-order upwind scheme



$$D_{xy} = u_{i,j}\frac{-\mathbf{u}_{i+2,j} + 8(\mathbf{u}_{i+1,j} - \mathbf{u}_{i-1,j}) + \mathbf{u}_{i-2,j}}{12h} + \frac{|u_{i,j}|h^3}{12}\frac{\mathbf{u}_{i+2,j} - 4\mathbf{u}_{i+1,j} + 6\mathbf{u}_{i,j} - 4\mathbf{u}_{i-1,j} + \mathbf{u}_{i-2,j}}{h^4} + v_{i,j}\frac{-\mathbf{u}_{i,j+2} + 8(\mathbf{u}_{i,j+1} - \mathbf{u}_{i,j-1}) + \mathbf{u}_{i,j-2}}{12h} + \frac{|v_{i,j}|h^3}{12}\frac{\mathbf{u}_{i,j+2} - 4\mathbf{u}_{i,j+1} + 6\mathbf{u}_{i,j} - 4\mathbf{u}_{i,j-1} + \mathbf{u}_{i,j-2}}{h^4}$$

Multi-directional finite-difference representation

- Discretization on the regular grid using a third-order upwind scheme



$$D_{\xi\eta} = \frac{1}{\sqrt{2}}\left(u_{i,j} + v_{i,j}\right)\frac{-\mathbf{u}_{i+2,j+2} + 8(\mathbf{u}_{i+1,j+1} - \mathbf{u}_{i-1,j-1}) + \mathbf{u}_{i-2,j-2}}{12\sqrt{2}h}$$

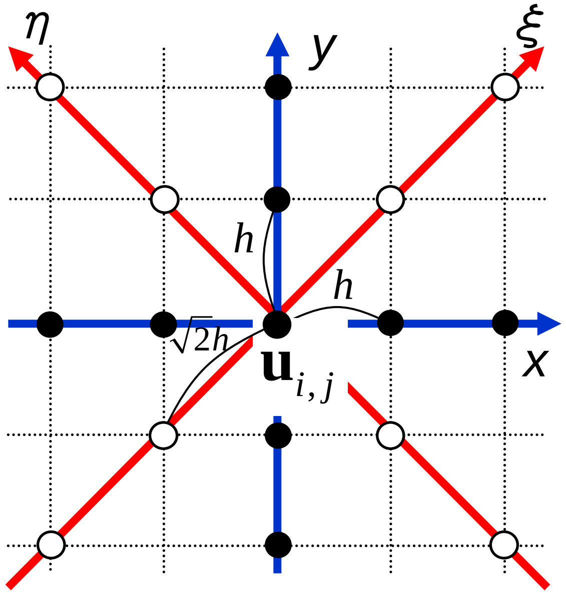
$$+\frac{1}{\sqrt{2}}\frac{\left|u_{i,j} + v_{i,j}\right|\left(\sqrt{2}h\right)^3}{12}\frac{\mathbf{u}_{i+2,j+2} - 4\mathbf{u}_{i+1,j+1} + 6\mathbf{u}_{i,j} - 4\mathbf{u}_{i-1,j-1} + \mathbf{u}_{i-2,j-2}}{\left(\sqrt{2}h\right)^4}$$

$$+\frac{1}{\sqrt{2}}\left(-u_{i,j} + v_{i,j}\right)\frac{-\mathbf{u}_{i-2,j+2} + 8(\mathbf{u}_{i-1,j+1} - \mathbf{u}_{i+1,j-1}) + \mathbf{u}_{i+2,j-2}}{12\sqrt{2}h}$$

$$+\frac{1}{\sqrt{2}}\frac{\left|-u_{i,j} + v_{i,j}\right|\left(\sqrt{2}h\right)^3}{12}\frac{\mathbf{u}_{i-2,j+2} - 4\mathbf{u}_{i-1,j+1} + 6\mathbf{u}_{i,j} - 4\mathbf{u}_{i+1,j-1} + \mathbf{u}_{i+2,j-2}}{\left(\sqrt{2}h\right)^4}$$

Multi-directional finite-difference representation

- Discretization on the regular and diagonal grids using a third-order upwind scheme



Difference formulas D_{xy} and $D_{\xi\eta}$ are mixed at ratio *r*.

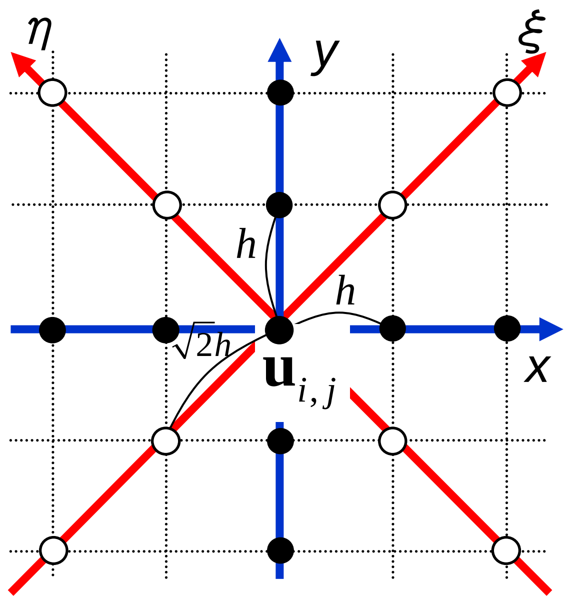
$$u\frac{\partial \mathbf{u}}{\partial x} + v\frac{\partial \mathbf{u}}{\partial y} \sim r \times D_{xy} + (1-r) \times D_{\xi\eta}$$

Ratio r of mix of D_{xy} and $D_{\xi\eta}$

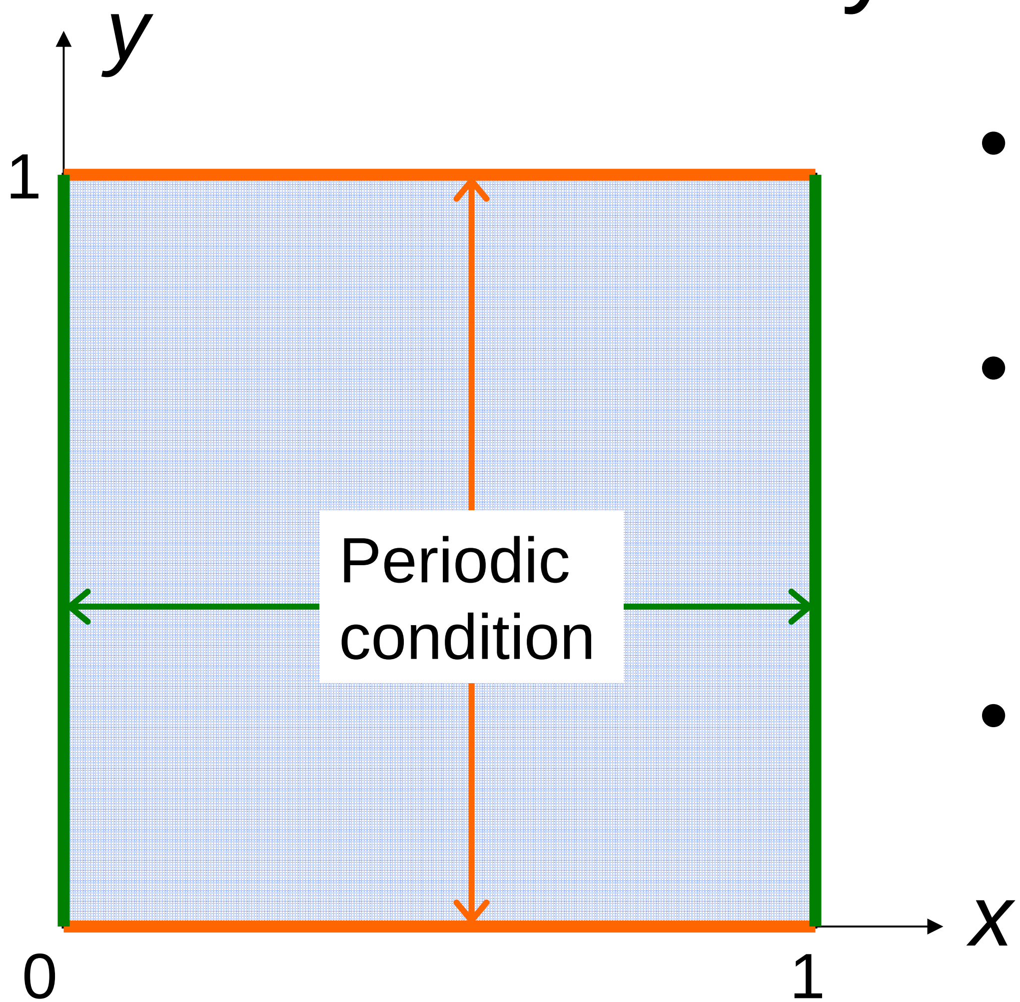
- In this paper, simulations are performed by changing the value of the ratio *r*.

$$u\frac{\partial \mathbf{u}}{\partial x}+v\frac{\partial \mathbf{u}}{\partial y} \sim \underline{r}\times D_{xy}+\underline{(1-r)}\times D_{\xi\eta}$$

r	
1	Regular difference approximation.
0.85	Weight of the diagonal grid is minimal (15%).
$2/3$	Highest accuracy in calculation of Laplace equation.
0	Difference approximation using only diagonal grid points.

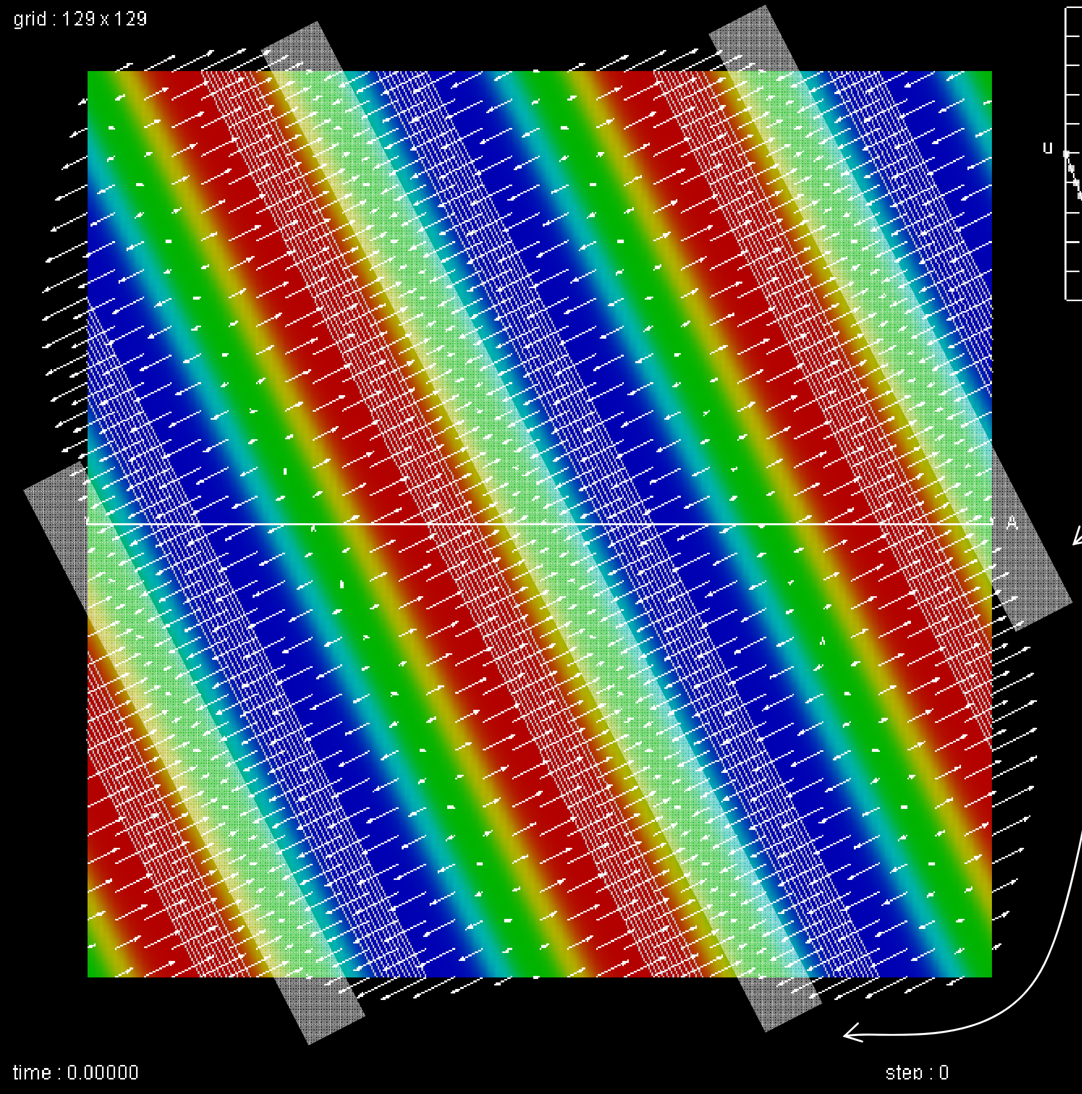


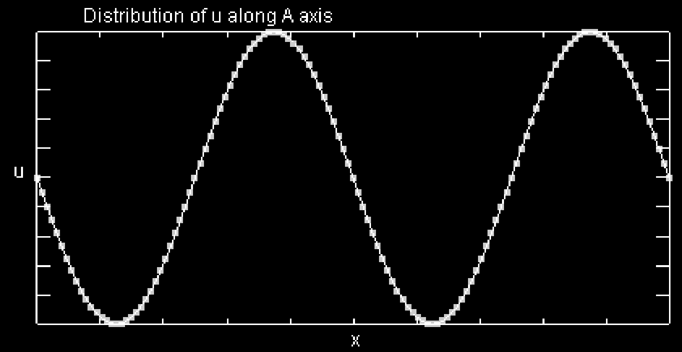
Computational domain and boundary conditions



- *x*-direction: periodic condition
- *y*-direction: periodic condition
- Computational grid: 129 x 129

Initial condition (flow I)



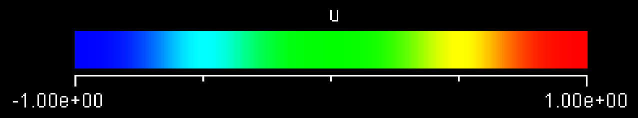


Discontinuities are formed here.

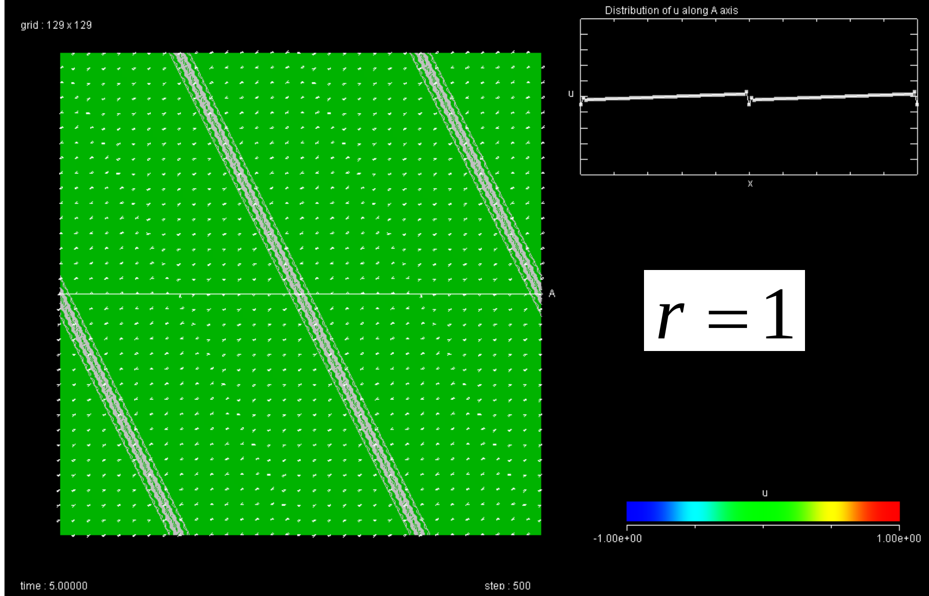
$$u(x,y) = \sin\left(4\pi(x + y/2)\right)$$

$$v(x,y) = \frac{1}{2}\sin\left(4\pi(x + y/2)\right)$$

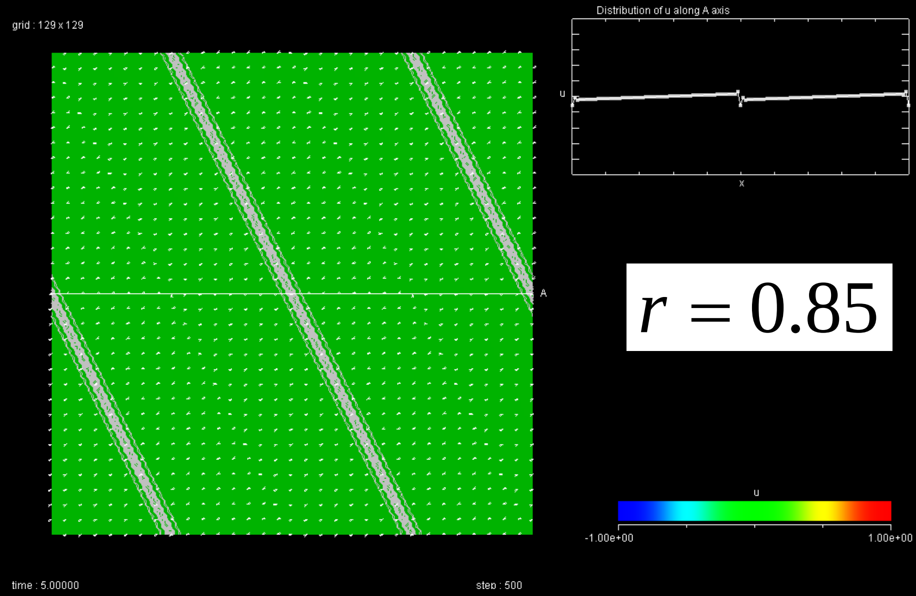
$$(0 \le x, y \le 1)$$



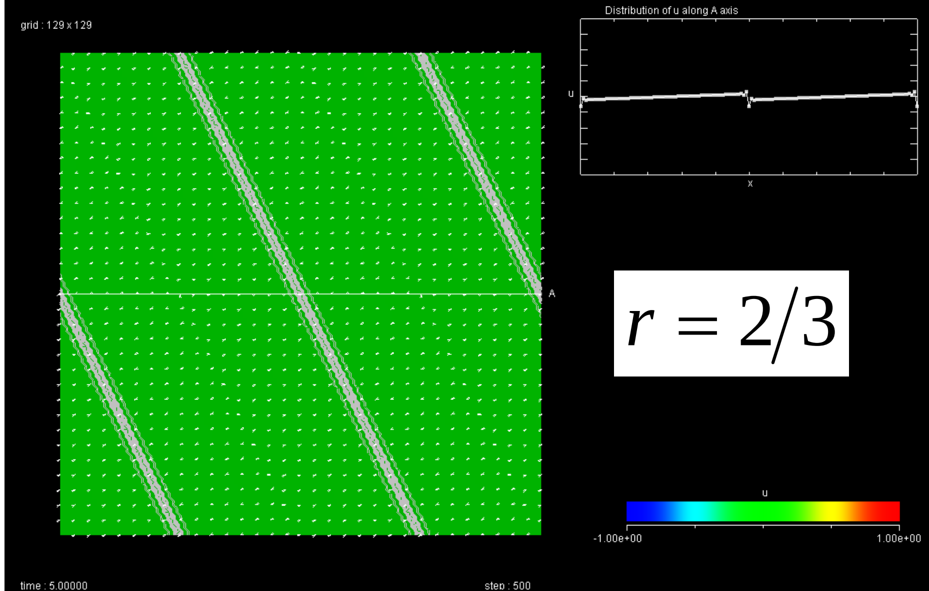
Computational results (flow I)



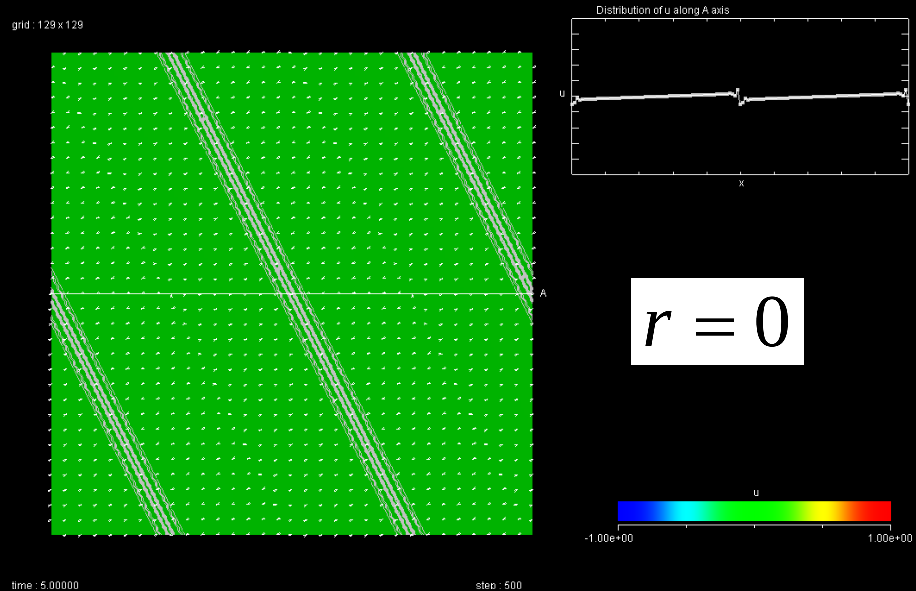
$r = 1$



$r = 0.85$

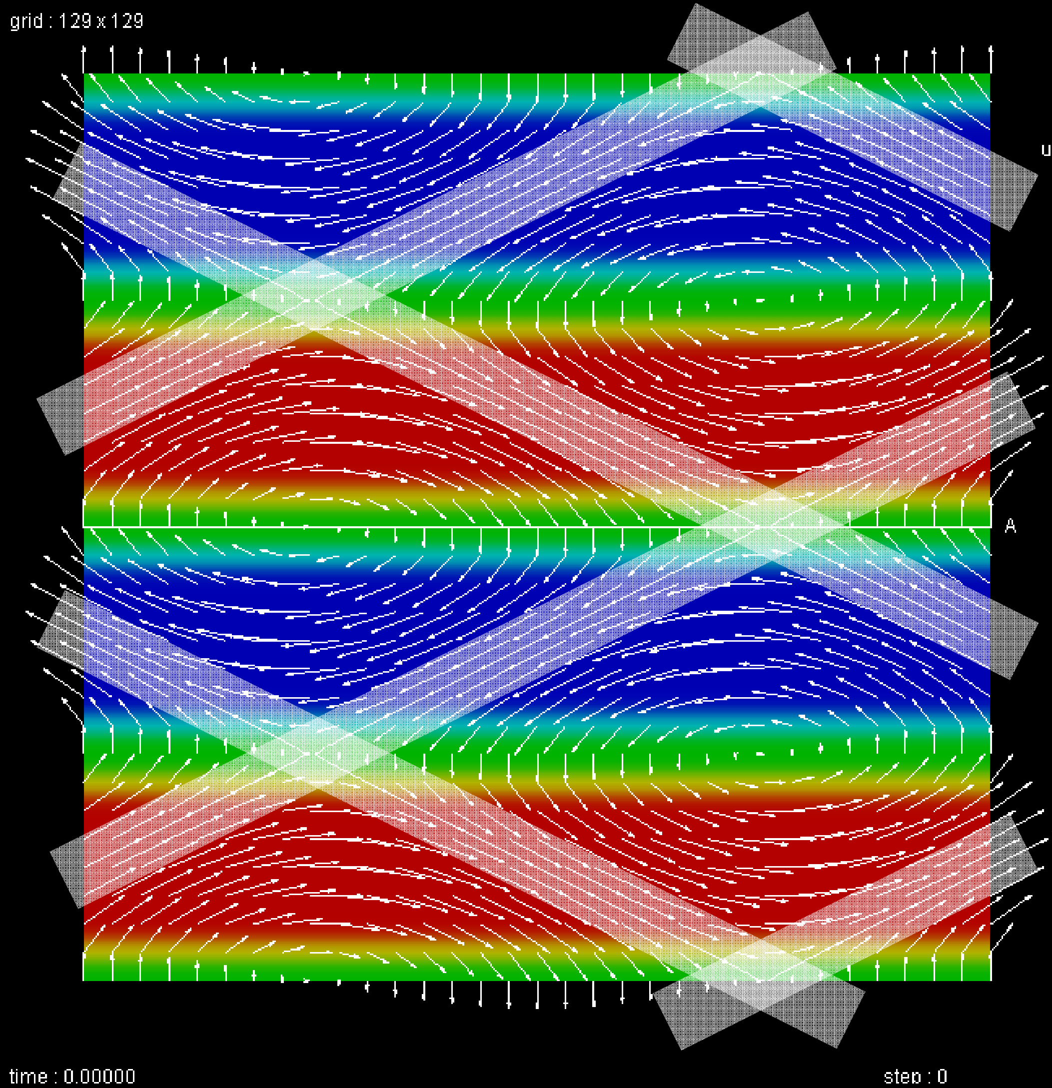


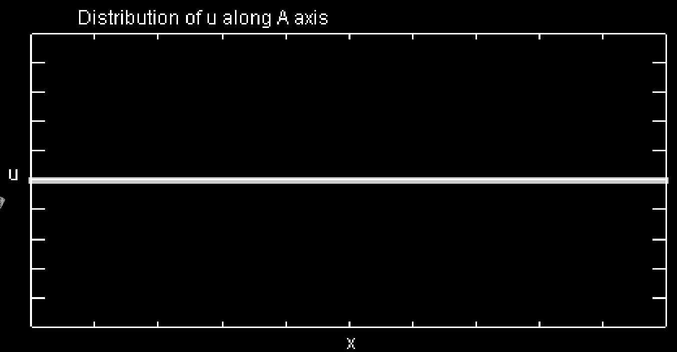
$r = 2/3$



$r = 0$

Initial condition (flow II)



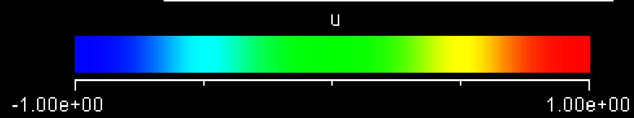


Discontinuities are formed here.

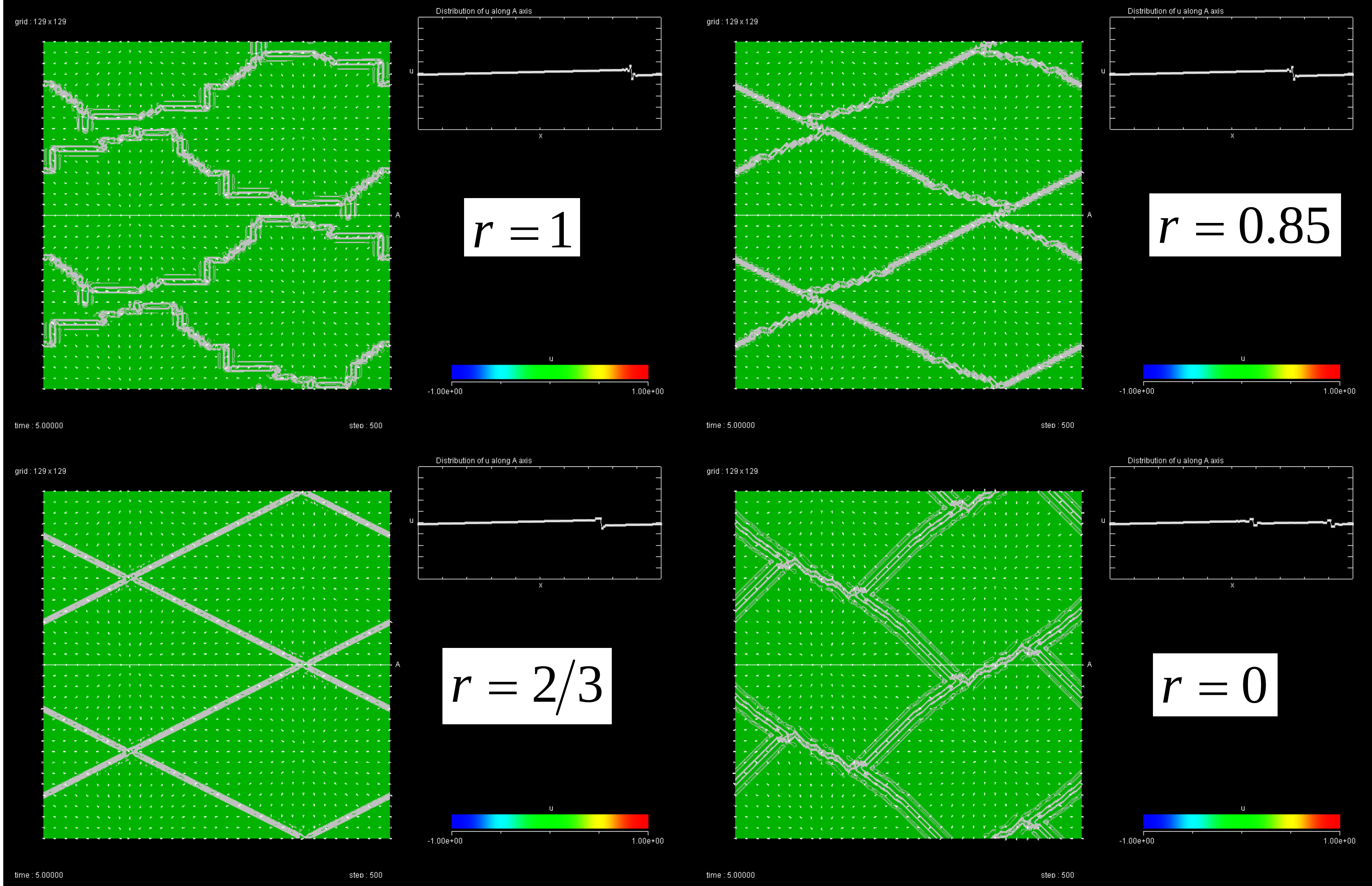
$$u(x,y)=\sqrt{\frac{8}{5}}\sin(4\pi y)$$

$$v(x,y)=\sqrt{\frac{2}{5}}\cos(2\pi x)$$

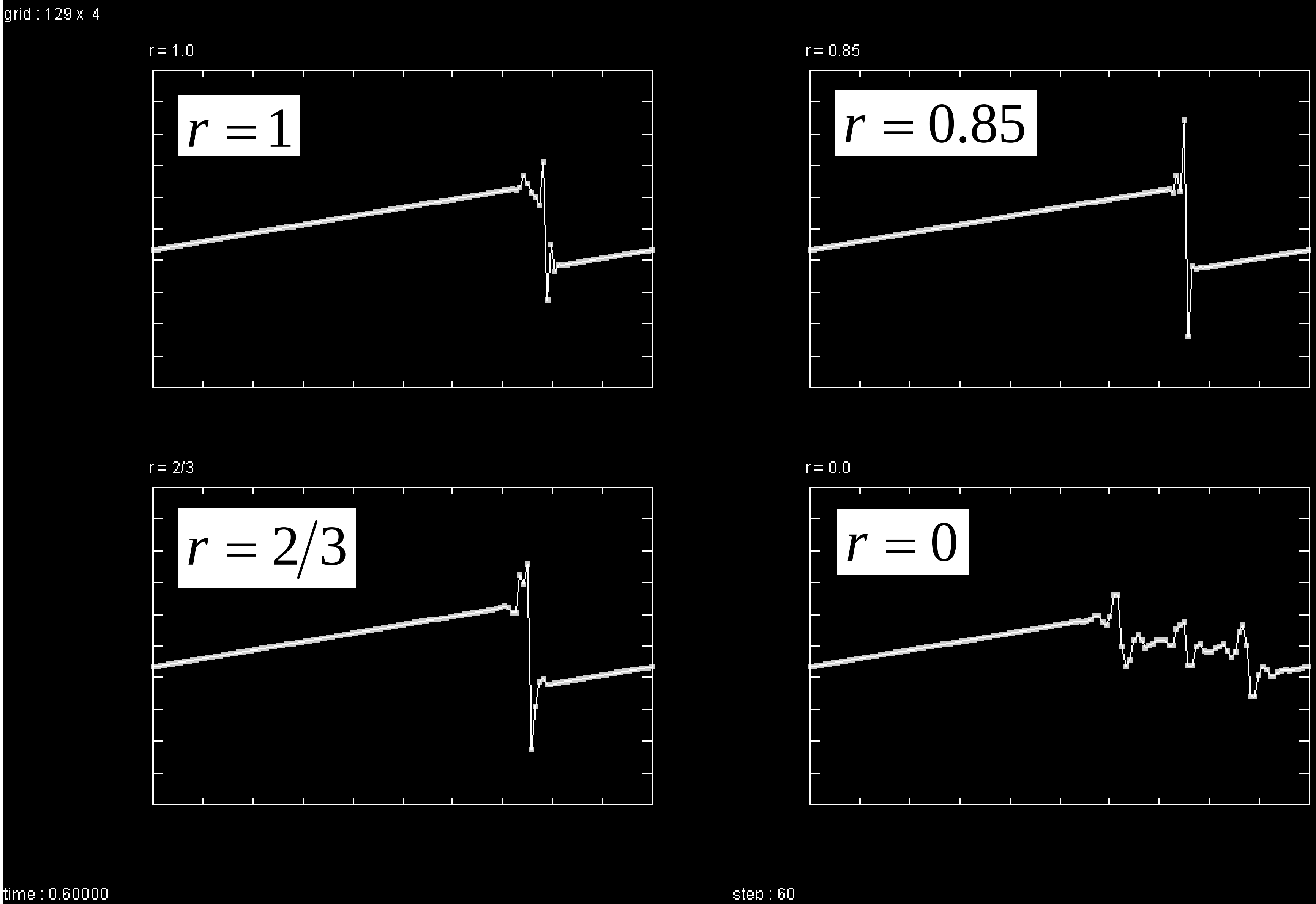
$$(0\le x,y\le 1)$$



Computational results (flow II)

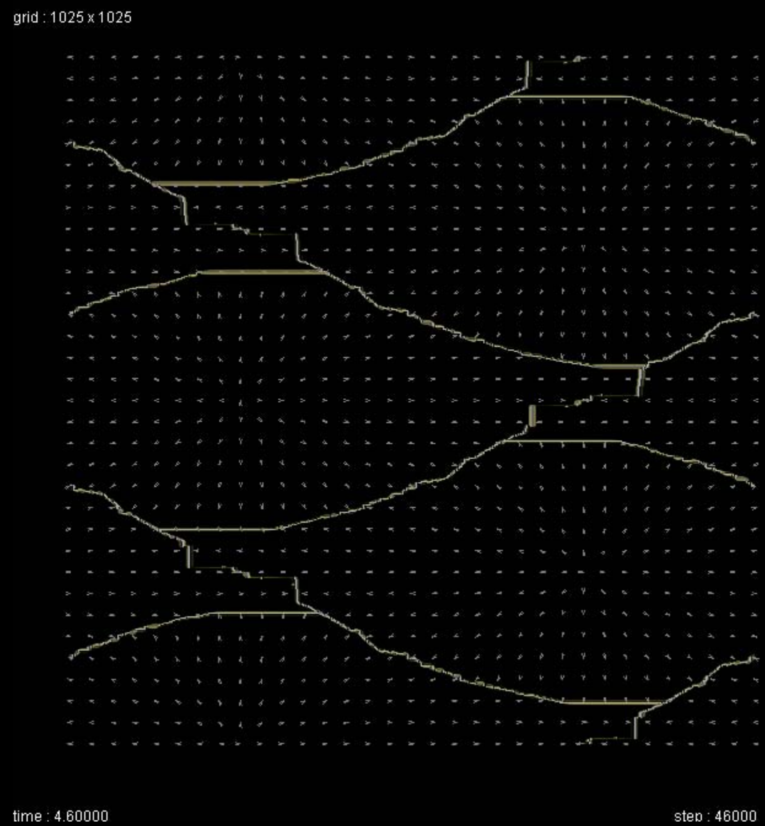


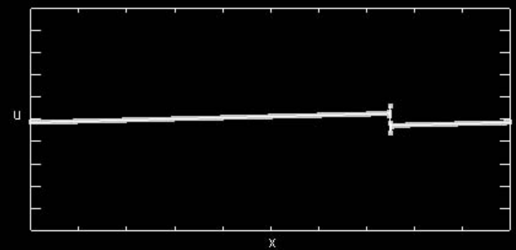
Computational results (flow II)



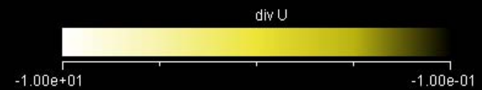
Computational results (flow II)

- High resolution computation
  - Finite-difference in regular coordinates
  - Grid size: 1024 x 1024
  - Third-order upwind scheme





$$r = 1$$



Conclusions

The multi-directional upwind scheme has the following advantages:

- Realization for any flow direction.
- Stable computation.
- Easy way to enable high accuracy computation.