Revisiting the issue of the rotational form for the convective term in the Navier-Stokes equation

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The aim of this talk is to review the recent development in the formulation of the approximation scheme for the convective term in the Navier-Stokes equations. The schemes commonly used in turbulent flow simulations are the discretization based on the rotation form, the convective form, the divergent form, and their combination. In this talk, the performance of these schemes are compared in the framework of the spectral, finite-difference (FDM) and finite-element methods (FEM).

The Navier–Stokes and continuity equations which describe the motion of incompressible fluid, which are derived via the conservation laws of mass and momentum within the small fluid volumes. When the Navier–Stokes equation is numerically discretized, it is known that, unless the discretized momentum and the kinetic energy are globally conserved, the result of the numerical simulation can give rise to instabilities; i.e., the conservative property of the Navier-Stokes equation should be retained in the numerically discretized scheme. The representative formulation for the convective terms which conserves the momentum and the kinetic energy is the skew-symmetric form [1]

$$\frac{1}{2}\left[\frac{\partial\left(u_j u_j\right)}{\partial x_i} + u_j \frac{\partial u_i}{\partial x_j}\right] \qquad (1)$$

and the rotational form [2]

$$u_j\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\frac{\partial\left(u_j u_j\right)}{\partial x_i} \qquad (2)$$

Deardorff [3] and Horiuti and Kuwahara [4] conducted a large-eddy simulation (LES) of a fully developed turbulent channel flow using the skew-symmetric form. Moin and Kim [2] conducted a LES of the same flow using the rotational form. In both studies, the FDM was used in the wall normal direction to approximate the partial derivatives, while either FDM or spectral method was used in the streamwise and spanwise directions. In these studies, certain differences in the turbulent statistics were found. The performance of the rotational and skew-symmetric forms was compared in Horiuti [5] in LES of the same flow. It was shown that a gradual decay of the turbulent state occurs when the rotational form is used, whereas good results are obtained when the skew-symmetric form is used.

Zang [6] reported extensive numerical experiments on the comparison of these two formulations in various turbulent flows using the spectral method. It was demonstrated that the skew-symmetric form gives fairly good results even in the presence of aliasing errors, whereas the rotational form performed poorly. The destabilizing effect of the aliasing errors in the rotational form was recognized in the wiggles of the contour plots, illustrating the vorticity distributions, but the decay of turbulence observed in [5] was not reported. Blaisdell *et al.* [7] presented a theoretical explanation as to why aliasing errors are reduced for the skew-symmetric form. Kravchenko and Moin [8] compared various formulations for the convective terms in LES of turbulent channel flow, in which the effect of the dealiasing for the convective terms was examined. They found that the difference between the results of the aliased and dealiased simulations was large for the rotational form, whereas it was minimal for the skew-symmetric form, confirming the results of Zang [6]. All these results consistently showed that the skew-symmetric form is superior to the rotational form.

More extensive analysis of the truncation error when the rotational form is used in the low-order accuracy FDM was carried out in Horiuti and Itami [7]. It was shown that the inaccuracy in the rotation form arises because the product rule

$$\frac{\partial(fg)}{\partial y} = \frac{\partial(f)}{\partial y} g + f \frac{\partial(g)}{\partial y} \qquad (3)$$

is not satisfied when the conventional low-order (second) finite difference scheme is used to approximate $\partial/\partial y$. Therefore, in the rotational form (2), the second and third terms in the right-hand side do not cancel out in the second order in terms of the grid spacing in the Taylor expansion, and yields a large truncation error. This truncation error induces the decay of turbulence when it is used in the channel flow, because it behaves similarly to the Coriolis force applied in the rotating channel.

Recently, there was a new twist in the analysis of the rotational form. William *et al.* [8] carried out an assessment in the framework of FEM. It was shown that the rotational form leads to a less accurate approximate solution than the usual convective form in FEM. The difference between the two forms is governed by the difference in the resolution of the Bernoulli and kinematic pressures, i.e., the under-resolution of the

Bernoulli pressure variable in the rotational form. Although, in general, the product rule is exactly satisfied when conforming or continuous elements are used in FEM, the Bernoulli pressure should be carefully treated when using the rotational form. In addition, the linear grad-div destabilization method which significantly reduces the influence of the pressure error on the velocity was proposed. In [8], the effectiveness of this method was demonstrated in the numerical experiments.

These studies demonstrated that the rotation form performs poorly in all three discretization methods, i.e., the spectral method, FDM and FEM.

[1] A. Arakawa, Computational design for long-term integration of the equations of fluid motion: two-dimensional incompressible flow, Part I, *J. Comput. Phys.* **1**, p.119 (1966).

[2] J. W. Deardorff, A numerical study of three-dimensional turbulent channel flow at large Reynolds numbers, *J. Fluid Mech.* **41**, p.453 (1970).

[3] P. Moin and J. Kim, Numerical investigation of turbulent channel flow, *J. Fluid Mech.* **118**, p.341 (1982).

[4] K. Horiuti and K. Kuwahara, Study of Incompressible Turbulent Channel Flow by Large-Eddy Simulation, *Proceedings of the 8th ICNMFD*, Aachen, West Germany, Springer Ed., (1982).

[5] K. Horiuti, Comparison of conservative and rotation forms in large eddy simulation of turbulent channel flow, *J. Comp. Phys.* **71**, p.343 (1987).

[6] T.A. Zang, On the rotation and skew-symmetric forms for incompressible °ow simulations, *Appl. Num. Math.* **7**, p.27 (1991).

[7] G. A. Blaisdell, E. T. Spyropoulos, and J. H. Qin, The effect of the formulation of nonlinear terms on aliasing errors in spectral methods, *Appl. Numer. Math.* **20**, 1 (1996).

[8] A. G. Kravchenko and P. Moin, On the effect of numerical errors in large eddy simulations of turbulent flows, *J. Comput. Phys.* **131**, 310 (1997).

[7] K. Horiuti and T. Itami, Truncation error analysis of the rotation form of convective terms in the Navier-Stokes equations, *J. Comp. Phys.*, **145**, p.671 (1998).

[8] L. William, M. Carolina C., N. Monika, O. Maxim, R. Leo G., On the accuracy of the rotation form in simulations of the Navier-Stokes equations, *J. Comp. Phys.,* **228**, p.3433 (2009).